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ALGORITHMS OF THE ASYMPTOTIC CONSTRUCTION OF LINEAR TWO-DIMENSIONAL THIN SHELL THEORY AND THE ST VENANT PRINCIPLE†

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A modified St Venant principle is formulated, governing the decay of the asymptotically dominant part of the stress–strain state due to a system of forces applied to an edge of a thin elastic body Four conditions for the satisfaction of the modified St Venant principle are derived and the possibility of using them to construct iterative processes for integrating the general equations of the theory of elasticity is established.

1. A shell is taken to be a thin three-dimensional body whose stress–strain state (SSS) is described by the static linear differential equations of the isotropic theory of elasticity. To fix our ideas we shall assume that the equations are taken with respect to the traditional triorthogonal coordinates

$$\mathbf{R} = \mathbf{r}(\beta_1, \beta_2) + \beta_3\mathbf{n}$$

and that the faces of the shell are given by the equations $\beta_3 = \pm\eta$ (where η is the half-thickness).

The main aim of this paper is to construct an iterative process for integrating the equations of the theory of elasticity, in which the first approximation should obey the theory for calculating thin shells.

Note that if the faces $\beta_3 = \pm\eta$ of the shell are not fixed, then its complete SSS composed of internal and boundary SSSs (boundary layers), localized near the edges or other stress concentrators of the shell. We accordingly observe the rule that is traditional in asymptotic methods in which one separates as much as possible the construction of the internal SSS from the construction of the boundary layers.

From a mathematical point of view this is justified by the fact that each type of SSS has its own features and requires its own mathematical techniques. From a physical point of view the separation is also rational because the practical value of data on the internal SSS and on the boundary layers are far from the same.

To achieve this separation it is natural to use the St Venant principle. However, as will be shown below, it cannot be applied in its canonical form.

This paper formulates a modified St Venant principle designed to be applied to the general theory of shells. It is not related to the total decay of the effect of the applied external forces, but only treats the "principal" decay, i.e. the decay of its asymptotically dominant component. An iterative process for integrating the equations of the theory of elasticity is discussed and coordinated with the modified St Venant principle.

2. An internal iterative process (giving the internal SSS) [1–3] can be implemented with a variable original accuracy, i.e. when constructing the original approximation an error of a different order is allowed. We will speak of iterative processes, constructed up to h^ρ, and call ρ the accuracy characteristic, if when deriving the leading approximation all terms of order $O(h^\gamma)$ with $\gamma > \rho$ are omitted in each separately taken equation. (An expression of $O(h^0)$ cannot always be taken to be an asymptotic error estimate.) The quantity h is taken to be a dimensionless half-thickness.

The most important internal iterative processes are constructed below with accuracy characteristic $\rho = 2 - 2q$ (where q is the variation index of the required SSS in the variables β_1, β_2).

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For such a ρ the differential equations of the internal iterative process in the leading approximation are equivalent to the Kirchhoff–Love equations [4]. We will use the latter since they are better known.

The iterative processes for the boundary layer are also described in the literature. For an accuracy characteristic $\rho = 1 - q$ its implementation reduces to the integration of differential equations for the anti-plane and plane problems of the theory of elasticity [1, 2, 4–7].

Thus the problem of the separated construction of differential equations in the theory of thin elastic bodies can be considered solved. We now turn to the separation of boundary conditions, i.e. to the problem of what boundary conditions should be assigned to the internal and boundary-layer differential equations. We will confine ourselves to the case when the stresses at the shell edge lying on coordinate surface $\beta_1 = 0$ are specified, there are no body forces, and stresses at the faces $\beta_3 = \pm\eta$ vanish.

Suppose, for example, that stresses σ_{1j} are specified at the $\beta_1 = 0$ edge. Then the corresponding three-dimensional boundary conditions can be conveniently expressed by the equations

$$\left[a\left(\sigma_{1j}^{(i)} + \sigma_{1j}^{(b)}\right)\right]_{\beta_1=0} = a_0 r_{1j} \quad (j = 1,2,3) \tag{2.1}$$

where $a = 1 + \beta_3/R_2$, $a_0 = a|_{\beta_1=0}$, $1/R_2$ is the normal curvature of the β_2-coordinate line, the superscripts i and b imply that the given quantity is related to the internal or boundary SSS, and r_{1j}, which define the boundary values of the stresses, are given functions of the variables β_2, β_3.

We shall assume that $\sigma_{ij}^{(i)}$ has been constructed by means of an internal iterative process with accuracy characteristic $\rho = 2 - 2q$, and we will express the corresponding results in terms of the Kirchhoff–Love theory, using the notation of [4]. This means that when $\beta_1 = 0$

$$a\sigma_{11}^{(i)} = \frac{T_1}{2\eta} - \beta_3\frac{3G_1}{2\eta^3}, \quad a\sigma_{12}^{(i)} = \frac{S_{21}}{2\eta} + \beta_3\frac{3H_{21}}{2\eta^3}, \quad a\sigma_{13}^{(i)} = -\frac{N_1}{2\eta}\frac{3}{2}\left(\beta_3^2 - \eta^2\right) \tag{2.2}$$

From this, using (2.1), we obtain

$$\left[\frac{T_1}{2\eta} - \beta_3\frac{3G_1}{2\eta^2} + a\sigma_{11}^{(b)}\right]_{\beta_1=0} = a_0 r_{11} \tag{2.3}$$

and two similar equalities which follow from the second and third formulae of (2.2).

We integrate (2.3) with respect to β_3 over the interval $(-\eta, +\eta)$ and obtain

$$T_1|_{\beta_1=0} + \int_{-\eta}^{+\eta}\left[a\sigma_{11}^{(b)}\right]_{\beta_1=0} d\beta_3 = \int_{-\eta}^{+\eta} a_0 r_{11} d\beta_3$$

From this we conclude that if the natural decay condition

$$\int_{-\eta}^{+\eta}\left[a\sigma_{11}^{(b)}\right]_{\beta_1=0} d\beta_3 = 0 \tag{2.4}$$

is assumed to be valid for the boundary SSS, then for the internal SSS we have the corresponding boundary condition

$$T_1|_{\beta_1=0} = \int_{-\eta}^{+\eta} a_0 r_{11} d\beta_3 \tag{2.5}$$

which is often applied in shell theory on the basis of physical considerations.

A correspondence of the form (2.3) $\leftrightarrow$ (2.5) also holds for all other decay conditions for a boundary SSS similar to condition (2.4).

Thus, if the boundary SSS of a shell obeys the canonical St Venant principle, then five conditions of the form (2.5) should be imposed on the internal SSS. This would lead to the well-known discrepancy with the order of the two-dimensional differential equations in thin shell theory. This is a consequence of the unjustified application of the canonical St Venant principle.

The separation of the boundary conditions in principle has been achieved. In particular, the boundary conditions for the two-dimensional differential equations of Kirchhoff–Love theory should be taken to be the relations that follow from the four correctly formulated decay conditions of the boundary SSS in the same way as (2.5) follows from (2.3). In order to obtain the edge conditions for the boundary SSS we must put

$$r_{1j} = r_{1j}^{(i)} + r_{1j}^{(b)} \quad (j = 1, 2, 3)$$

and assume that

$$\left[a\sigma_{1j}^{(i)}\right]_{\beta_1=0} = a_0 r_{1j}^{(i)}, \quad \left[a\sigma_{1j}^{(b)}\right]_{\beta_1=0} = a_0 r_{1j}^{(b)} \tag{2.6}$$

Assuming that the internal SSS has already been constructed, one can find $r_{1j}^{(i)}$ from the first equality in (2.6), and consequently also determine $r_{1j}^{(b)}$. As a result the second quality in (2.6) determines the required edge conditions for the boundary SSS.

In Section 3 we formulate a modified St Venant principle. It describes the "dominant" decay and the four conditions which ensure such a decay. Using this principle the solution of the boundary-value problem of the theory of elasticity for thin bodies can be constructed as follows.

1. Define the internal SSS of the shell to be the original approximation of the internal iterative process with accuracy characteristic $\rho = 2 - 2q$, i.e. construct it using the equations of Kirchhoff–Love theory.

2. In the boundary SSS construct just the asymptotically dominant part, which decays.

3. Of the three edge conditions of the original problem, formulated in terms of the three-dimensional theory of elasticity derive, using the above scheme, the four boundary conditions for the Kirchhoff–Love theory equations and the three edge conditions for the boundary SSS.

4. Using the conditions of item 3 sequentially construct the original approximations for the internal SSS and the decaying part of the boundary SSS, which together constitute the original approximation of the required SSS.

The formulations of the above boundary-value problems for internal and boundary differential equations are based on approximation arguments. Moreover, the scheme does not assume the construction of an asymptotically secondary non-decaying part of the boundary layer. All these errors can be formally removed by means of iteration.

The two-dimensional Kirchhoff–Love theory occupies a central place in the above scheme. It enables one to investigate approximately a not-too-strongly varying SSS of a shell sufficiently far away from its boundary or other concentrators of the required SSS. (Note that the scheme described does not anticipate the case when the Kirchhoff–Love solution vanishes identically.)

This apparently contradicts the literature which often refers to inadequacies inherent in Kirchhoff–Love theory and to the paradoxes which it somehow creates (see, e.g., [8–10]). However in all the refuting examples the theory is applied to cases for which it would not be used according to the above definition. These are primarily problems in which explicit or disguised elastic boundary phenomena play a major part. Moreover, the refuting examples do not take into account the requirement that the variability of the required SSS is not excessive. We know that Kirchhoff–Love theory cannot be applied in the opposite situation. It is also inapplicable in the well-known "paradoxical" problem of the torsion of a strip, where at the corners of the strip the required SSS has infinite variability.

In terms of the above approximation scheme one can, if necessary, also investigate the boundary SSS of the shell. This can be performed as a second stage of the calculation and reduces to solving the differential equations governing the decaying part of the boundary layer.

We interpret our non-traditional approximation scheme for shell investigation by means of the following dual model for a thin elastic body. It can be thought of as the combination of a shell-model for the internal SSSs and a family of strip-models for the decaying part of the boundary layer.

The shell-model is the middle surface of a thin body, endowed with elastic properties corresponding to the Kirchhoff–Love hypotheses.

The family of strip-models that represents the boundary layer need only be constructed near the edge. Assuming that the latter lies along the $\beta_1 = 0$ coordinate surface, we define each strip to be a part of the $\beta_2 =$ const coordinate surface adjacent to the edge. The variable β_2 is the family parameter of the strip-models. We determine the elastic properties of the strips by assuming that their stresses and strains satisfy the complete system of equations of the three-dimensional theory of elasticity and that the stresses σ_{2j} ($j = 1, 2, 3$) can be treated as interaction forces between the strips, whose form has to be chosen in the proper manner.

In order for the SSSs of the shell-strips to decay, the edge forces $\sigma_{1j}|_{\beta_1=0}$, must obey a number of requirements compatible with those for the shell-model, i.e. that number must be four.

If one assumes, for example, that all the interaction forces in the family of strip-models vanish, the decay conditions transform into the five requirements of the canonical St Venant principle. This returns us to the previous disparity between the orders of the two-dimensional shell-model equations and the number of boundary conditions they must obey. The disparity is removed if in the strip-model differential equations we take into account the pairing-off of tangential stresses, and assume that the tangential interaction forces σ_{21}, σ_{23} are non-zero and that only the normal force σ_{22} vanishes.

3. We will carry out calculations exemplifying the arguments of Section 2 for the case when a three-dimensional thin shell degenerates into the semi-infinite layer $G\{\alpha_1 \leqslant -\infty < \alpha_2 < +\infty; -h \leqslant \alpha_3 \leqslant +h\}$ where the (α_1, α_2, α_3) are Cartesian coordinates.

We will explain below why this does not lessen the generality of the final conclusions.

We shall assume that the layer G satisfies the homogeneous face conditions

$$\sigma_{3k}\big|_{\alpha_3=\pm h} = 0 \quad (k = 1,2,3) \tag{3.1}$$

and the inhomogeneous edge conditions

$$\sigma_{1k}\big|_{\alpha_1=0} = r_{1k}(\alpha_1,\alpha_3) \quad (k = 1,2,3) \tag{3.2}$$

Together they mean that for the layer G we consider an SSS that has been caused by forces **R** with components r_{1k} acting at the edge $\alpha_1 = 0$.

Thus, in the dual model of Section 2, the shell-model is taken to be a semi-infinite layer, and the family of strip-models is a collection of semi-infinite strips $g(\alpha_2) = \{\alpha_1 \geqslant 0; -h \geqslant \alpha_3 \leqslant +h\}$ whose position at the edge is specified by the parameter α_2.

We begin with the linear equations of the isotropic theory of elasticity in Cartesian coordinates and write them as follows:

$$\begin{aligned} &D_i\left(P;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)+d_i\left(Q;\frac{\partial}{\partial\alpha_2}\right)=0 \quad (1\leqslant i\leqslant 3)\\ &D_j\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)+d_j\left(P;\frac{\partial}{\partial\alpha_2}\right)=0 \quad (4\leqslant j\leqslant 9)\end{aligned} \tag{3.3}$$

(Henceforth i and j are always taken to be indices having the values indicated here.) In (3.3) the SSS of the layer G has been split into two terms

$$P = (\sigma_{12}, \sigma_{23}; \upsilon_2), \quad Q = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{13}; \upsilon_1, \upsilon_3) \tag{3.4}$$

The letters P and Q within the parentheses of (3.3) show which of the required quantities occur in the given term on the left-hand side. Symbols indicating the derivatives to be found in that term are shown after the semi-colon.

It is assumed that the equations for the layer G have been reduced to dimensionless form: the stresses σ_{mn} ($m, n = 1, 2, 3$) are in terms of Young's modulus, the displacements υ_m are in terms of the half-thickness η, and α_1, α_2, α_3 h are Cartesian coordinates and the half-thickness in terms of some characteristic length L.

The actual differential expressions D_i, d_i, D_j, d_j in (3.3) are

$$D_1\left(P;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\sigma_{21}}{\partial\alpha_1}+\frac{\partial\sigma_{23}}{\partial\alpha_3},\quad d_1\left(Q;\frac{\partial}{\partial\alpha_2}\right)=\frac{\partial\sigma_{22}}{\partial\alpha_2}$$

$$D_2\left(P;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\upsilon_2}{\partial\alpha_1}-2(1+\nu)\sigma_{12},\quad d_2\left(Q;\frac{\partial}{\partial\alpha_2}\right)=\frac{\partial\upsilon_1}{\partial\alpha_2}$$

$$D_3\left(P;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\upsilon_2}{\partial\alpha_3}-2(1+\nu)\sigma_{23},\quad d_3\left(Q;\frac{\partial}{\partial\alpha_2}\right)=\frac{\partial\upsilon_3}{\partial\alpha_2}$$

$$D_4\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\sigma_{11}}{\partial\alpha_1}+\frac{\partial\sigma_{13}}{\partial\alpha_3},\quad d_4\left(P;\frac{\partial}{\partial\alpha_2}\right)=\frac{\partial\sigma_{12}}{\partial\alpha_2} \tag{3.5}$$

$$D_5\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\sigma_{31}}{\partial\alpha_1}+\frac{\partial\sigma_{33}}{\partial\alpha_3},\quad d_5\left(P;\frac{\partial}{\partial\alpha_2}\right)=\frac{\partial\sigma_{32}}{\partial\alpha_2}$$

$$D_6\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\upsilon_1}{\partial\alpha_1}-[\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})],\quad d_6=0$$

$$D_7\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=-[\sigma_{22}-\nu(\sigma_{11}+\sigma_{33})],\quad d_7\left(P;\frac{\partial}{\partial\alpha_2}\right)=\frac{\partial\upsilon_2}{\partial\alpha_2}$$

$$D_8\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\upsilon_3}{\partial\alpha_3}-[\sigma_{33}-\nu(\sigma_{11}+\sigma_{22})],\quad d_8=0$$

$$D_9\left(Q;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=\frac{\partial\upsilon_1}{\partial\alpha_3}+\frac{\partial\upsilon_3}{\partial\alpha_1}-2(1+\nu)\sigma_{13},\quad d_9=0$$

4. We will treat Eqs (3.3)–(3.5) as strip-model equations, represent the grouped variables in them as sums

$$P=P'+P'';\quad Q=Q'+Q'' \tag{4.1}$$

and require P' and Q' to be the decaying part of the boundary layer, and P'' and Q'' to be the remaining non-decaying part. In Section 5 it is shown that this can be achieved if P' and Q' are governed by the equations

$$D_i\left(P';\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=0\quad(1\leqslant i\leqslant 3) \tag{4.2}$$

$$D_j\left(Q';\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)=-d_j\left(P';\frac{\partial}{\partial\alpha_2}\right)\quad(4\leqslant j\leqslant 9) \tag{4.3}$$

and P'' and Q'' obey the questions

$$D_i\left(P'';\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)+d_i\left(Q'';\frac{\partial}{\partial\alpha_2}\right)=-d_i\left(Q';\frac{\partial}{\partial\alpha_2}\right)\quad(1\leqslant i\leqslant 3) \tag{4.4}$$

$$D_j\left(Q'';\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right)+d_j\left(P'';\frac{\partial}{\partial\alpha_2}\right)=0\quad(4\leqslant j\leqslant 9)$$

The systems (4.2)–(4.4) are of course collectively equivalent to system (3.3). In order for the boundary equations (3.1) and (3.2) to be satisfied also, it is necessary to impose the conditions

$$\sigma'_{3k}=0\quad(\alpha_3=\pm h);\quad \sigma'_{1k}=r_{1k}\quad(\alpha_1=0) \tag{4.5}$$

$$\sigma''_{3k}=0\quad(\alpha_3=\pm h);\quad \sigma''_{1k}=0\quad(\alpha_1=0),\quad k=1,2,3 \tag{4.6}$$

on P', Q' and P'', Q''.

Here and henceforth it is assumed that the method described in Section 2 for separating the boundary conditions has already been employed and that r_{1k} is taken to be the part of the stresses specified at the edge $\beta_1 = 0$ which corresponds to the decaying part of the boundary SSS. (It is of course assumed that the r_{1k} have also been reduced to dimensionless form in a suitable manner.)

5. We will now consider the formulation of a modified St Venant principle for the SSS P', Q'. Three out of the nine equations of system (4.2), (4.3) obeyed by P' and Q' express the balance of forces on a differential element of the strip-model. These equations can be written in expanded form as

$$X_1=\frac{\partial\sigma'_{11}}{\partial\alpha_1}+\frac{\partial\sigma'_{12}}{\partial\alpha_2}+\frac{\partial\sigma'_{13}}{\partial\alpha_3}=0,\quad X_2=\frac{\partial\sigma'_{21}}{\partial\alpha_1}+\frac{\partial\sigma'_{23}}{\partial\alpha_3}=0$$
$$X_3=\frac{\partial\sigma'_{31}}{\partial\alpha_1}+\frac{\partial\sigma'_{32}}{\partial\alpha_2}+\frac{\partial\sigma'_{33}}{\partial\alpha_3}=0 \tag{5.1}$$

The following boundary-value problem must be solved for the system of equations (4.2), (4.3) (it is appropriate to call them the decaying SSS equations): in an infinite half-strip g construct a solution satisfying conditions (4.5) at the faces of $\alpha_3 = \pm h$ and at the edge $\alpha_1 = 0$ and decaying at infinity.

We shall assume that the decay is as strong as required in subsequent arguments. The boundary-value problem has a solution only when certain necessary conditions, expressing the balance of the external forces on the entire strip g, are satisfied. They will be identified with necessary and sufficient conditions for the modified St Venant principle to be applicable.

The equalities

$$\int X_1dg=\int X_2dg=\int X_3dg=\int(\alpha_3X_1-\alpha_1X_3)dg=0 \tag{5.2}$$

follow in an obvious manner from (5.1), the range of integration being the domain occupied by the strip g.

We will use the following techniques, usually employed in the theory of elasticity, when transforming these integrals we change the order of integration over α_1, α_3, using integration by parts with respect to one or other of α_1 and α_3, and transforming the integrands using equalities (5.1).

Moreover, we will use the face and edge conditions (4.5), and also postulate decaying properties. Using all this we can transform the first equality of (5.2) as follows:

$$\int_{-h}^{+h}d\alpha_3\int_0^\infty\frac{\partial\sigma'_{11}}{\partial\alpha_1}d\alpha_1+\frac{\partial}{\partial\alpha_2}\int_{-h}^{+h}d\alpha_3\int_0^\infty\sigma'_{12}d\alpha_1+\int_0^\infty d\alpha_1\int_{-h}^{+h}\frac{\partial\sigma'_{13}}{\partial\alpha_3}d\alpha_3=0 \tag{5.3}$$

In the second term on the left-hand side we perform integration by parts with respect to α_1. We obtain

$$\frac{\partial}{\partial\alpha_2}\int_{-h}^{+h}d\alpha_3\int_0^\infty\sigma'_{12}d\alpha_1=\frac{\partial}{\partial\alpha_2}\int_{-h}^{+h}[\alpha_1\sigma'_{12}]_0^\infty d\alpha_3-\frac{\partial}{\partial\alpha_2}\int_{-h}^{+h}d\alpha_3\int_0^\infty\alpha_1\frac{\partial\sigma'_{12}}{\partial\alpha_1}d\alpha_1$$

Here the first term on the right-hand side is zero from the assumption that σ'_{12} decays sufficiently rapidly as α_1 increases. In the second term one can replace $\partial\sigma'_{12} / \partial\alpha_1$ by $-\partial\sigma'_{23} / \partial\alpha_3$, using the second equality of (5.1). Hence, performing the integration over α_3 and using the face conditions (4.5), we again obtain zero. This means that the second term on the left-hand side of (5.3) vanishes.

In a similar manner, having performed the integration over α_3 and used the face conditions (4.5), we conclude that the third term on the left-hand side of (5.3) also vanishes. Consequently

$$\int X_1 dg = \int_{-h}^{+h} d\alpha_3 \int_0^{\infty} \frac{\partial\sigma'_{11}}{\partial\alpha_1} d\alpha_1$$

Hence, using the edge conditions (4.5) and the decaying conditions for $\alpha_1 = \infty$, we have the first condition for satisfying the modified St Venant principle

$$\int_{-h}^{+h} r_{11} d\alpha_3 = 0 \tag{5.4}$$

After similar calculations two other conditions are obtained from the second and third equalities in (5.2)

$$\int_{-h}^{+h} r_{12} d\alpha_3 = \int_{-h}^{+h} r_{13} d\alpha_3 + \frac{\partial}{\partial\alpha_2} \int_{-h}^{+h} r_{12}\alpha_3 d\alpha_3 = 0 \tag{5.5}$$

and the fourth equality of (5.2) gives

$$\int_{-h}^{+h} \alpha_3 r_{11} d\alpha_3 - I = 0, \quad I = 2\int \frac{\partial}{\partial\alpha_2}(\alpha_3\sigma'_{12}) dg \tag{5.6}$$

Relations (5.4)–(5.6) are also the four required conditions for the existence of a decaying SSS P', Q' generated by the edge forces r_{1k}.

In Section 7 we will show that P', Q' exceed P'', Q'' in the well-known asymptotic sense. Hence one can take the properties of P' and Q' that are obtained here to be a modified St Venant principle. It amounts to the fact that the requirements (5.4)–(5.6) are the "dominant" decaying conditions for that SSS which is generated by the edge forces. We emphasize that the modified St Venant principle, unlike the canonical one, agrees with the dual model of a thin elastic body (Section 2), because for the shell-model it only requires four boundary conditions to be substituted into it. From the physical point of view the reduction of the number of decay conditions is explained by the fact that the modified St Venant principle is directed towards a family of strip-models in which shear stresses are not excluded from the α_2 = const faces. Hence it is unnecessary to impose a fifth condition (i.e. the absence of a torque at the edge) in order to obtain St Venant decay.

6. The meaning of the first three decay conditions (5.4)–(5.6) is obvious. They imply the three well-known boundary conditions of two-dimensional plate theory

$$T_1 = S_{21} = N_1 + \partial H_{21} / \partial s = 0$$

We will discuss the integral term I of Eq. (5.6).

We transform this quantity assuming, as was discussed in Section 4, that r_{1k} in the second equation of (4.5) satisfies the conditions of the modified St Venant principle.

To calculate I one needs to know the stress σ'_{12} throughout the domain g, i.e. to have, for a fixed β_2, the decaying solution of the anti-plane problem in the half-strip g with face and edge conditions (4.5). Here, if we restrict ourselves to the asymptotically dominant parts of the solution, σ'_{12} is to be understood as the solution of the anti-plane problem satisfying conditions (4.5) with $k = 2$.

Because σ'_{12} satisfies, by hypothesis, the decay condition when $\alpha_1 = \infty$, the problem in question has a solution in a very general case. It is simply constructed using trigonometric Fourier series [4, 11].

Substituting this solution into the expression for I and integrating over α_1, α_3, we obtain the required result. For specified β_2 it gives a correction to the quantity G_1 in the corresponding boundary condition in Kirchhoff–Love theory. In the general case this relation is expressed as a numerical series which depends on the parameter β_2.

In the specific and not too complex cases that appear in most problems of practical importance, the final result can also turn out to be simple.

Suppose that in the second edge condition (4.5)

$$r_{12} = A + B\alpha_3, \quad A = \frac{S}{2E\eta}, \quad B = \frac{3H}{2E\eta^2} \tag{6.1}$$

where A and B are given dimensionless functions of α_2, the quantities S and H are the boundary shear force and the boundary torque, and η is the actual half-thickness.

Substituting (6.1) into the first decay condition (5.5) we obtain $S = 0$. Consequently, $A = 0$.

We denote by $\Sigma_{12}(\alpha_1, \alpha_3)$ the shear stress given by the decaying solution to the anti-plane problem in the half-strip g with the condition

$$\Sigma_{12} = \alpha_3 \quad (\alpha_1 = 0)$$

Because the above anti-plane problem is linear and its formulation does not depend on β_2, we have $\sigma'_{12} = B\Sigma_{12}$. Hence one can write

$$I = 3I_0 \frac{1}{EL^2}\frac{\partial H}{\partial \alpha_2}, \quad I_0 = \frac{1}{h^2}\int_0^\infty d\alpha_1 \int_{-h}^{+h} \Sigma_{12}\alpha_3 d\alpha_3$$

(where L is the length-scale referred to in Section 3 and defined by the formulae $\eta = Lh$).

By the method described above we obtain

$$I_0 = \frac{384}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} \approx 0.4200$$

(calculations in other notation are given in [4, p. 465]).

The first term of (5.6) can in an obvious way be reduced to

$$\int_{-h}^{+h} r_{11}\alpha_3 d\alpha_3 = \frac{1}{EL^2} G|_{\alpha_1=0}$$

Hence, omitting the common factor $(E\eta^2)^{-1}$, we obtain, instead of (5.6), the so-called reduced boundary condition for two-dimensional plate and shell theory

$$G + 1.2600h\partial H / \partial\alpha_2 = 0$$

Thus, in the case under consideration, all four decay conditions for the modified St Venant principle correspond to boundary conditions described in the literature of the Kirchhoff–Love theory (see, for example, [4]). They also take into account the classical Kelvin–Tait correction to the shearing force and the correction to the bending moment G, first observed when constructing the so-called reduced boundary conditions in [11]. (The present paper shows that this result is not general: strictly speaking, it is only true for cases when the edge force has the form (6.1).)

7. We now compare the asymptotic orders of the quantities P', Q' and P'', Q''. The main importance of this question is that the answer will tell us whether it is possible to use the modified St Venant principle in the general theory of shells in the same way as the canonical principle is used in the strength of materials, i.e. whether there is any meaning to two-dimensional shell theory. Below, without any pretence at rigour, we give arguments relevant to this problem.

To determine P' and Q' it is necessary to solve Eqs (4.2) and (4.3) using boundary conditions (4.5). The formulation of these boundary conditions in explicit form does not depend on the small parameter h. However, P', and Q' are by assumption decaying solutions, and it is natural to expect that they will vary exponentially with α_1 with large (as $h \to 0$) exponents. We therefore make the variable scale-change traditional in such situations by putting

$$\alpha_1 = h\xi_1, \quad \alpha_2 = h^\kappa \xi_2, \quad \alpha_3 = h\xi_3, \quad \upsilon_k = hw_k \quad (k = 1,2,3) \tag{7.1}$$

and postulating that differentiation with respect to ξ_k no longer changes the asymptotic order of the required functions.

The powers of h in (7.1) are chosen using the following arguments: the power of h in front of ξ_3 corresponds to the well-known fact that near the edge the variability index with respect to α_3 for the shell boundary layer is equal to unity. The same power should occur in the formula for ξ_1: only in that case does one obtain equations describing boundary layers whose asymptotically dominant terms are identical with those in the equations of the plane and anti-plane problems of the theory of elasticity. This agrees with intuitive representations of boundary layers and is found to be in complete agreement with the nature of the boundary-value problems governing P' and Q'. The quantity κ in the formula for α_2 is taken to be a specified number $(0 \leqslant \kappa \leqslant 1)$. It is the variability index with respect to the variable α_2 of the quantities r_{11}, r_{12}, r_{13} in the edge conditions (3.2). The index of h in the last formula in (7.1) will be discussed below.

Introducing (7.1) into (3.3) and (3.5), we obtain formulae for changing to new independent variables for D_i, D_j, d_i, d

$$D_\rho\left(R;\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\alpha_3}\right) = h^{\lambda-1}D_\rho\left(\overline{R};\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_3}\right)$$

$$d_\rho\left(R;\frac{\partial}{\partial\alpha_2}\right) = h^{\lambda-\kappa}d_\rho\left(\overline{R};\frac{\partial}{\partial\xi_2}\right) \tag{7.2}$$

$\lambda = 0$ when $\rho = 1, 4, 5$; $\lambda = 1$ when $\rho = 2,3,6,7,8$

Here the subscript ρ can take values of i or j, R can be taken to be either P or Q, and the bar over the R means that in (3.4) the displacements υ_k should be replaced by w_k, i.e. instead of (3.4) one should use the formulae

$$\overline{P} = (\sigma_{12},\sigma_{23};w_2), \quad \overline{Q} = (\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{13};w_1,w_3) \tag{7.3}$$

Using (7.2) and (7.3), we can write system (4.2), (4.3) in the form

$$D_i\left(\overline{P};\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_3}\right) = 0 \quad (1 \leqslant i \leqslant 3) \tag{7.4}$$

$$D_j\left(\overline{Q}';\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_3}\right) = -h^{1-\kappa}d_j\left(\overline{P}';\frac{\partial}{\partial\xi_2}\right) \quad (4 \leqslant j \leqslant 9) \tag{7.5}$$

Specifying $\overline{Q}'$ by the formula

$$\overline{Q}' = \overline{Q}_0' + h^{1-\kappa}\overline{Q}_1' \tag{7.6}$$

we obtain for $\overline{Q}_0'$, $\overline{Q}_1'$ the equations

$$D_j\left(\overline{Q}_0';\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_3}\right) = 0 \tag{7.7}$$

$$D_j\left(\bar{Q}_1';\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_3}\right)=-d_j\left(\bar{P}';\frac{\partial}{\partial\xi_2}\right)\quad(4\leq j\leq 9)\tag{7.8}$$

The first of these should be solved using the edge conditions

$$\sigma_{11}'=r_{11},\quad \sigma_{13}'=r_{13}\quad(\xi_1=0)$$

and to the second one should add the appropriate homogeneous edge conditions.

Thus, for $\bar{P}'$ $\bar{Q}_0'$, $\bar{Q}_1'$, we obtain boundary-value problems in which the small parameter h does not occur. Furthermore, we note that in all the above problems it is only the differential equations that are inhomogeneous, or the edge boundary conditions. We interpret these inhomogeneities as external (loading or deformational) effects and make use of the following conditional notation

$$\bar{P}'\Rightarrow[r_{12}],\quad \bar{Q}_0'\Rightarrow[r_{11},r_{13}],\quad \bar{Q}_1'\Rightarrow\left\{d_j(\bar{P}';\partial/\partial\xi_2)\right\}\tag{7.9}$$

These mean that the quantities on the right of the arrows are the cause of the appearance of the stresses or displacements on the left of the arrows. Square brackets are used in [7.9] in those cases when the "external effects" are at the edge, i.e. applied to the coordinate line $\xi_1=0$, and curly brackets are used when the "external effects" are applied to a two-dimensional segment adjacent to $\xi_1=0$. (We recall that here we are talking about quantities with a single prime, i.e. possessing St Venant decay.)

The equations governing P'' and Q'' are briefly described by (4.4). Expanding the latter using (3.5) we find that in expanded form they are a complete inhomogeneous set of elasticity equations. Its inhomogeneity is represented by the terms $d_i(Q',\partial/\partial\alpha_2)$, i.e. it depends only on the quantity Q', which can be considered to be known, assuming that the decaying part of the SSS has already been constructed.

System (4.4) should be solved using the homogeneous boundary conditions (4.6). Hence one can again assume that the corresponding SSS has been generated by "external effects" and write yet another relation of the form (7.9)

$$(P'',Q'')\Rightarrow\{d_j(Q';\partial/\partial\alpha_2)\}\tag{7.10}$$

As the basis of the following arguments we assume, without any pretence at rigour, that not too far away from the edge the asymptotic orders of the various SSSs can be identified with the order of the generating "external effects". Here, however, we take into account that in (7.9) and (7.10) the near-edge (in curly brackets) "external effects" have the St Venant form, i.e. they decay exponentially with an exponent of order h^{-1}. Hence, to relations (7.9) and (7.10), we add the conditional equality

$$\{A\}=h[A]\tag{7.11}$$

This means that if the "external effects" at the edge have asymptotically identical intensities, then in asymptotic discussions the near-edge "external effects" must be used with an additional factor of h.

The formulation of the boundary-value problem for P'' and Q'' does not depend on the small parameter h. Furthermore, no St Venant decay is assumed for P'' and Q''. Hence for relations (7.10) there is no need to resort to the first and third scale-transformation formulae; it is sufficient to use the second. This means that of the transformation formulae of the form (7.2) we now only need to use one

$$d_i\left(Q'';\frac{\partial}{\partial\alpha_2}\right)=h^{-\kappa}d_i\left(Q';\frac{\partial}{\partial\xi_2}\right)$$

Relation (7.10) is accordingly replaced by the relation

$$(P'',Q'') \Rightarrow h^{1-\kappa}[d_j(Q';\partial/\partial\xi_2)] \qquad (7.12)$$

When comparing the asymptotic orders of P', Q' and P'', Q'' using the superposition principle, we decompose the general case into two (which corresponds to decoupling into plane and anti-plane boundary layers).

Case 1: $r_{12} \sim h^0$; $(r_{11}, r_{13}) = 0$.

Case 2: $r_{12} = 0$; $(r_{11}, r_{13}) \sim h^0$.

Here ~ denotes asymptotic commensurability, and the zero exponent for h was chosen to fix our ideas.

In Case 1, using the rough assumption made above, it follows from (7.2), (7.6), (7.9) and (7.12) that

$$P' \sim h^0, \quad Q_0' = 0, \quad Q_1' \sim h^{1-\kappa}, \quad (P',Q') \sim h^0$$

from which relation (7.12) gives

$$(P'',Q'') \sim h^{2-2\kappa}$$

In the same way we find in Case 2 that

$$(P',Q') \sim h^0, \quad (P'',Q'') \sim h^{1-\kappa}$$

(These arguments refer only to stresses, so that the difference between $\bar{P}$, $\bar{Q}$ and P, Q is irrelevant and the bars have therefore been dropped.)

It has thus been shown that the use of a modified St Venant principle can be justified in shell theory just as the canonical principle can be used in the strength of materials. Here the necessity to change over to so-called "reduced boundary conditions" should not be considered as a defect of the Kirchhoff–Love theories, but as a reflection of the details of the modified St Venant principle.

In [12, 13] the so-called shearing theories of plates and shells were compared with two-dimensional theories obtained asymptotically. We will continue this discussion using the results obtained here.

We can assume that the dual model for a thin elastic body that we have proposed corresponds to a dual system of "shearing hypotheses". Part of this follows from the properties of the shell-model carrying the internal SSS. It is related to the assumption of a linear distribution of displacements along the thickness (which is not explicitly stated, but is used in a significant manner because, as has been shown in [4], the variational principles which lie at the heart of shearing theories depend on such properties).

The other part of the "shearing hypotheses" reflects the properties of the family of strip-models carrying out the boundary SSS. This includes the assumption of the tangential direction of the interaction forces between the strips. It gives rise to the need, when investigating the boundary SSS, to take into account the shears, because otherwise the properties of the two parts of the dual model will not "match". However, it remains unclear whether taking account of the shears is sufficient for the correct construction of the boundary SSS. It is also unreasonable to assume that the refinement of the internal SSS can be achieved merely by taking into account the shear only. Furthermore, "shearing hypotheses" are considered to be standard for the internal and boundary SSS, although there is no reason why taking into account the shear is a priority not just for the boundary, but also for internal SSS. From the results of [12, 13] it follows that this is simply not the case.

In conclusion we note two properties.

1. The replacement of the shell by a semi-infinite layer G in Section 3 does not reduce the generality of the conclusions we have reached, although it leads to a considerable simplification in the calculations. The latter are essentially identical with those which would have been performed in the general tri-orthogonal coordinate system and are described in detail in [4].

2. Also, for simplicity, this paper has considered only the case when all the stresses on the edge are specified, i.e. there are no initially unknown reaction forces at the edge clampings. Techniques for

finding the reduced boundary conditions and, consequently, for solving the problems raised here when there is edge clamping, are described in [4, 11].

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